Twisted Square

Driving Question

How can congruence, similarity, and other rules of geometry be used to determine—without measurement—the area of a square constructed using four segments within a larger square?

Background

Before computers or other types of technology made it easy to produce exact copies of images, people made use of clever mechanical devices such as the pantograph to copy or rescale drawings or images. A pen used for drawing or tracing over the original drawing is linked to a second pen by an arrangement of hinged parallelograms. Depending on the arrangement of the hinged parallelograms, the original drawing may be copied at the same scale, producing a second, congruent image. Or, the original can be copied at a smaller or larger scale and thus resized so that the second image is similar to the original but not congruent.

A Hoberman Sphere, with mechanical hinges similar to the pantograph, is a special type of 3-dimensional structure that can expand or contract while retaining its spherical shape. Some Hoberman Spheres are displayed as exhibits in museums and science centers, and use motorized mechanisms to expand and contract their hinges, increasing or decreasing their diameter while they maintain similarity. Smaller, plastic Hoberman Spheres are made as toys.

The pantograph and Hoberman Sphere illustrate how similarity is related to scale and congruence. In this project you will produce a scaled and transformed square and investigate the properties of congruence and similarity that give rise to this smaller, "twisted" square.

Project Objectives

In this project you will work with the members of your group to:

- construct a series of larger squares with smaller, twisted squares inscribed in them according to a sequence of instructions.
- experiment with geometric transformations of triangles, squares, and other polygons in the plane.
- describe the rotations and reflections that carry a triangle or square onto itself.
- given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using graph paper, tracing paper, or geometry software.
- determine the area of composite two-dimensional figures comprised of a combination of triangles, trapezoids, and other regular polygons.

- prove the congruence of two or more polygons visually.
- use trigonometric ratios and the Pythagorean equation to solve right triangles.
- make connections between the mathematical concepts in this project and real life applications.
- present your results and conclusions to your class.

Materials and Equipment

Mandatory Equipment

- Graph paper, several sheets
- Construction paper, several sheets
- Colored pencils
- Ruler or straightedge
- Scissors
- Tape

Optional Equipment

- Dynamic geometry software
- Drawing compass

Key Concepts for Background Research

Research your project topic based on the driving question above. Use any resources available to research background information that will help you to complete your project.

Below is a list of key concepts that may be helpful when doing your background research.

- Transformation (rotation, translation, reflection)
- Slope of parallel lines
- Slope of perpendicular lines
- Midpoint of a segment
- Opposite side (of an angle)
- Adjacent side (of an angle)
- Pythagorean equation for right triangles
- Similarity
- Scale factor
- Angle-Angle Theorem for proving similarity
- Ratio of similarity
- Congruence
- Sine of an acute angle (in a right triangle)
- Cosine of an acute angle (in a right triangle)
- Vertex (plural – vertices)
- Inscribed
- Tessellation
- Quadratic equation
- Roots or solutions of a quadratic equation

Safety and Maintenance

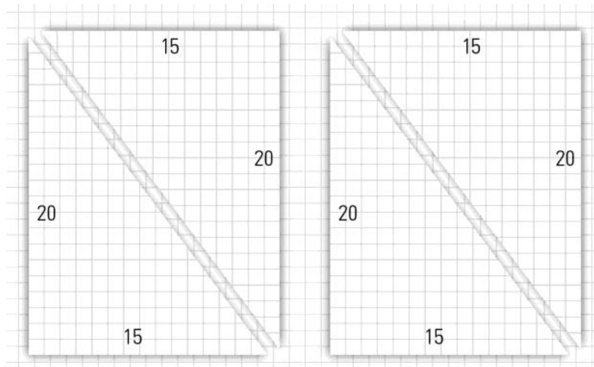
Add this important safety precaution to your normal classroom procedures:

- Handle all sharp objects carefully, including drawing compasses, scissors or craft/hobby knives.

Investigation

Use these guiding questions to help answer the driving question:

1. Use graph paper to draw and cut out four right triangles whose perpendicular sides are 15 units and 20 units respectively. You may find it helpful to draw the 15-unit sides in one color, the 20-unit sides in a second color, and the hypotenuse in a third color:



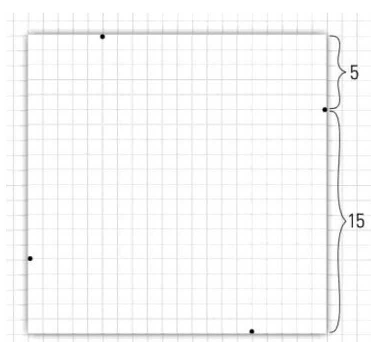
2. Arrange the four right triangles to form a square with sides each 35 units in length. Without measuring, how can you determine the length of one side of the inner square? What is its length?

3. In the previous step you used four right triangles to form the largest square possible. Rearrange the four right triangles to form a smaller square whose sides are less than 35 units in length, without overlapping the triangles. What is the length of a side of the twisted inner square? How can you determine this without measuring the side?

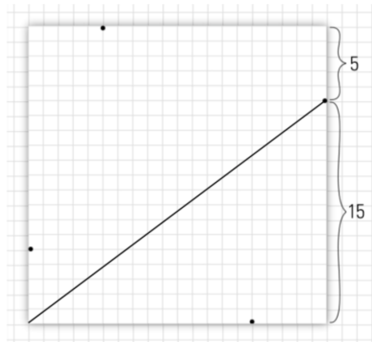
4. Form an even smaller twisted square by rearranging the triangles again. This time overlap the four right triangles to form a square whose sides are 20 units in length. Can the length of the inner square's side be found using the same method as for the two previous inner squares? Justify your answer.

5. Draw and cut out a square that is 20 units on each side. This square is congruent to the square in the previous step.

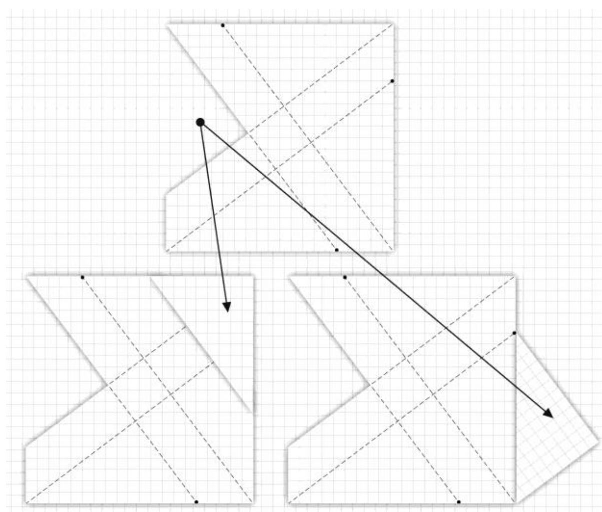
6. Use one of the right triangles to divide each side of the 20 x 20 square into two segments – one segment of 15 units, one segment of 5 units. Mark a point on each side to indicate this division:



7. Connect each vertex of the square to the point on the opposite side by drawing a segment along the hypotenuse of the right triangle:



8. Cut out one or more of the small triangles. Later in this activity you will derive the area of the "twisted square" formed in a construction similar to this. Part of the derivation requires you to apply the following skills: calculating acute angles in triangles, and solving for unknown side lengths in right triangles. Practice these skills within this construction.



a. Using colored pencils identify the opposite and adjacent sides of one of the acute angles in the 15 unit x 20 unit right triangles.

b. To find the acute angles of the right triangle, which of the following trigonometric relationships do you need to apply: sine, cosine, or tangent?

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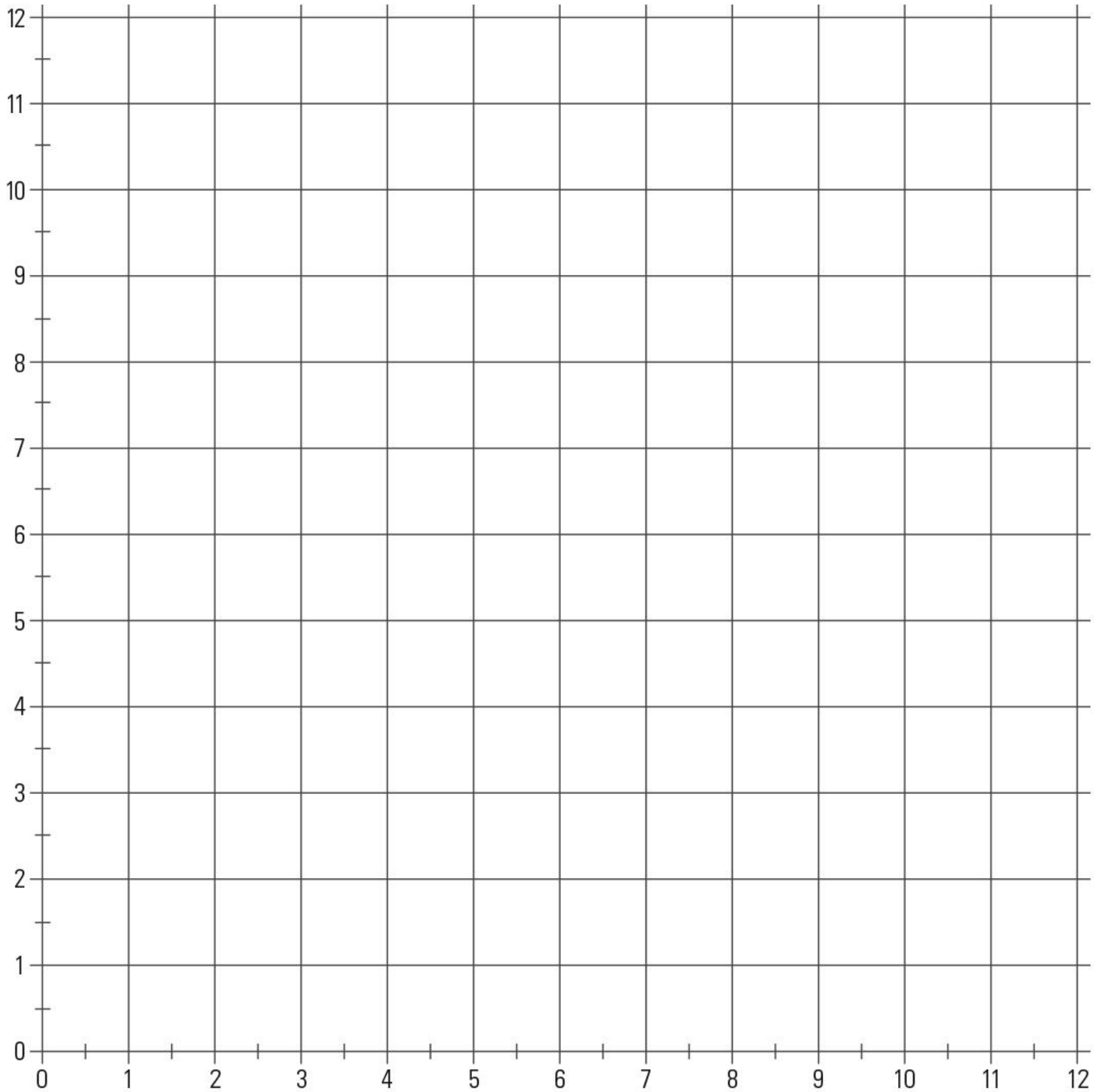
c. Calculate the acute angles and show your steps.

d. How many degrees are in the measure of the third angle of the right triangle? Show the steps of your calculation.

e. Use the grid squares to verify your calculations.

9. Use the small triangle(s) to prove visually that the inner twisted square is actually a square. Describe or draw your method.

10. Define a square ABCD, with side length of 10 units. Name the midpoints M, N, P, R, of sides AB, BC, CD, and DA, respectively. Draw the segment that joins vertex A to midpoint N. Do this for the remaining vertices and midpoints. Use the grid provided.



11. What other types of polygons have been formed inside the square ABCD as a result of constructing the smaller square?

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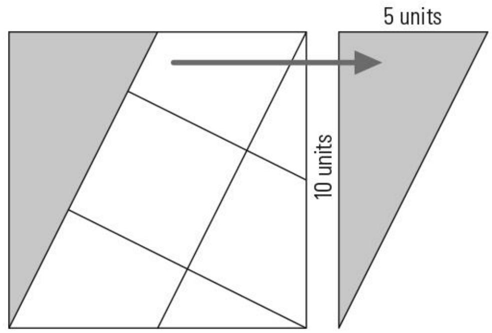
12. What is the length, in units, of the segment that connects the larger square's vertex to the midpoint on the opposite side? Show your method and calculations for determining the length.



13. Describe at least one way in which one of these polygons can be transformed to create a new square congruent to the "twisted" square. Use additional paper to trace images if necessary.

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14. Use transformations to construct additional squares congruent to the smaller square, to form a 5-square cross that is similar to a "plus sign" or the symbol on the flag of Switzerland. Use additional paper to trace images if necessary.

15. Using additional graph paper onto which you trace the original square ABCD and any transformations, prove visually that all of the triangles created by transformations are congruent both to one another and to the small triangles within the large square ABCD.

16. Using additional graph paper as needed, prove visually that all of the smaller squares are congruent.

17. The segment length calculated in Question 12 forms the hypotenuse of a right triangle. What is the number in degrees in the acute angles of this triangle? How can you determine the angle without measuring the angle directly?

18. What theorem can you apply to prove that two triangles are similar? Apply this theorem to prove that any one of the larger triangles in square ABCD is similar to any of the smallest triangles you constructed within square ABCD.

19. Show two different ways to find the area of the smaller twisted square. Verify your results by shading in the grid squares to sum the total area.

Synthesis Questions

The answers to the following questions will guide you to further critical thinking about your project topic, and will build knowledge to help answer questions that your classmates may ask when you present.

1. How does the definition of congruent polygons allow you to prove congruence visually? What are some possible shortcomings of this kind of proof?

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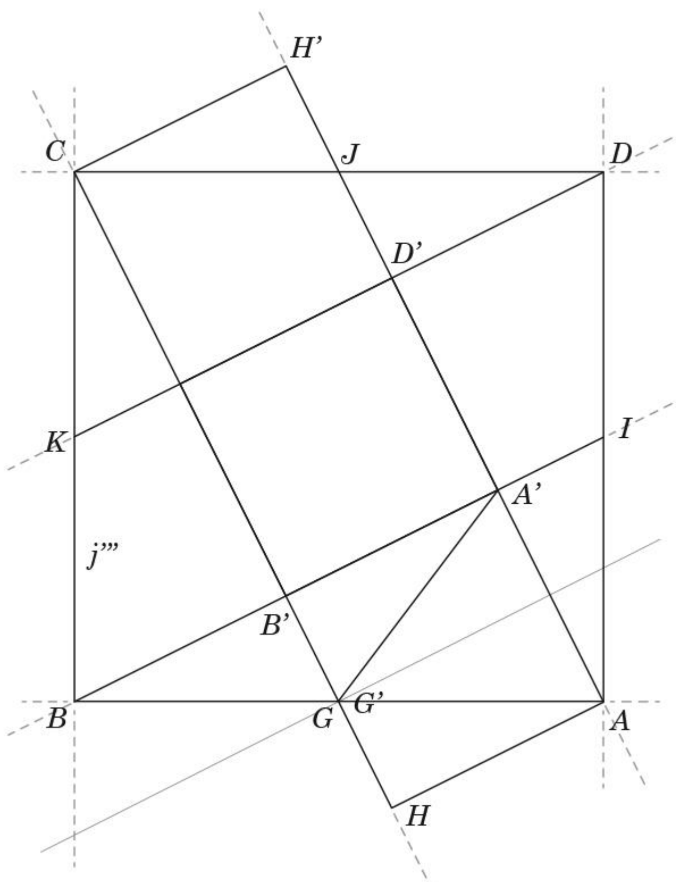
2. If a classmate unfamiliar with your work on this project saw the diagram below and stated that one of the angles formed by the segments appeared to be obtuse (for example ∠BGA'), how could you verify without using a protractor to measure that the apparent obtuse angle contains more than 90º?



3. **Integers that satisfy the Pythagorean equation for right triangles are called "Pythagorean triples." There are infinitely many such sets of integers, but the 3-4-5 right triangle is the smallest Pythagorean triple. The triangles you drew and used in the first part of this project were multiples of the 3-4-5 right triangle. How can you confirm this?**

4. **How can you prove algebraically that pairs of adjacent segments that form the smaller square are, in fact, perpendicular and that opposite pairs of segments are, in fact, parallel? Use additional graph paper as needed.**

Extension and Real Life Application

The answers to the following questions will help you make connections between the concepts relevant to your project topic and their applications to real world problems, and will further add to your knowledge to help answer questions that your classmates may ask when you present.

1. **Discuss at least one real life application (and extension) of concepts relevant to this entire project.**

2. **Create a set of four congruent right triangles that satisfy the Pythagorean triple of 5-12-13, and use these triangles to construct a small inner square as you did in investigation Questions 1 - 10. What is the area, in square units, of the inner square?**

3. **Suppose a right triangle has its shortest side 8 units in length. If the dimensions of the perpendicular side and the hypotenuse complete a Pythagorean triple, what would be the area, in square units, of an inner square constructed as in the previous question?**

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4. **If you constructed the square from the right triangles in the previous question, what percent of the area of the largest possible square is the area of the smallest possible square you can make using the methods followed in this project?**

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5. **The ancient Egyptians used ropes knotted at regular intervals to measure right angles. Three ropes were connected and each rope had a different number of knots tied at regular intervals. When pulled taut, a right triangle was formed which could be used to ensure that a stone block had each face perpendicular to the faces adjoining it. How many knots would have been tied in each rope? Explain your thinking.**

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6. **Suppose you have eight congruent right, scalene triangles with sides of length a, b, and c. Show how you can use four of the triangles to create two congruent rectangles of length a and width b, and another four of the triangles to create a square with sides length c. How can these figures be used to demonstrate the Pythagorean Theorem?**

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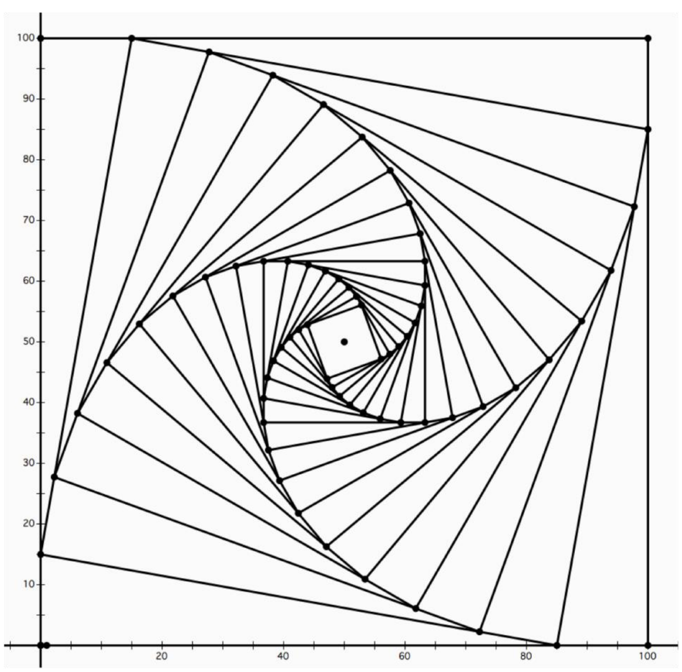
7. **Suppose you can construct a point anywhere on the side of the larger square and connect that point to the opposite vertex. This point divides the side into segments of length a and b. Create the smaller, inner square in this manner by constructing congruent points and segments on each side of the larger square. What must be the ratio of the lengths a and b such that the area of the smaller square is one-half that of the larger square?**

8. An artist wants to create a sculpture based on multiple rotations of a square tile, stacked one on top of the other. She plans to make a series of squares that are successively smaller, and rotate them by some fixed amount as she stacks each new layer. She has seen a 2-dimensional design on a Cartesian coordinate system with the vertices of a square at points (0, 0), (100, 0), (100, 100), and (0, 100) respectively. She observes that a rotated inner square was formed by plotting the vertex of an inscribed square at points (0, *a*), which is some distance above the origin. Another vertex of this inscribed square has been plotted at (*a*, 100). This defines a right triangle whose short leg has length *a*, whose long leg has length *b*, and whose hypotenuse is one side of the inscribed square. What must be the ratio of *a* to *b* so that each new square can be rotated by a fixed amount when she stacks it on top of the one below it? What is the ratio of similarity for the squares?



<table>
<tr><td colspan="2">Mathematics Investigation Report – Assessment Rubric
Level I</td><td colspan="5">Name:
Math Class:</td></tr>
<tr><td>Component</td><td>To receive highest marks the student:</td><td>4
Expert</td><td>3
Prac-titioner</td><td>2
Appren-tice</td><td>1
Novice</td><td>0
No Attempt</td></tr>
<tr><td>1. Preparation and Research</td><td>☐ Collaborated with group members on relevant research
☐ Brought and prepared all items necessary for their presentation
☐ Prepared supporting material for their presentation
☐ Prepared a thorough presentation for their classmates</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>2. Demonstrations, Models, or Experiments</td><td>☐ Obtained all necessary materials and used them to explore the guiding questions of the investigation
☐ Completed and documented results, data, and observations for guiding questions of the investigation
☐ Performed experiments or demonstrations proficiently for others, or explained clearly a model and the concepts the student investigated with the model</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>3. Content</td><td>☐ Presented written and spoken explanations that were mathematically accurate and paraphrased in the student’s own words
☐ Answered all questions posed by their teacher or classmates correctly and thoughtfully
☐ Answered project synthesis and real-world application questions correctly and thoroughly
☐ Communicates clearly an understanding of the connection between their model or experiments and the driving question and theory behind the over-arching concept</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>4. Real-World Application and Extension</td><td>☐ Identifies in written and spoken explanations the application of the topic to the real world, including specific examples
☐ Thoroughly discusses relevance of the topic to real life</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>5. Collaboration and Contribution</td><td>☐ Group members equitably shared responsibility for:
• Research of guiding questions
• Presentation and discussion of results</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>6. Presentation</td><td>☐ Delivered their content clearly and thoroughly, in an organized, logical manner with
• Eye contact and poise
• Appropriate voice level and clarity
• Addressing the class/audience</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="6">Total Points for Investigation (Maximum of 24 Points)</td><td></td></tr>
</table>

Guidelines for Marks:

4 = **Expert:** Distinguished command of the topic; students show insightful and sophisticated communication of their understanding
3 = **Practitioner:** Strong command of the topic; students show reasonable and purposeful communication of their understanding
2 = **Apprentice:** Moderate command of the topic; students show adequate but basic communication of their understanding
1 = **Novice:** Partial command of the topic; students show limited and insufficient communication of their understanding
0 = **No Attempt**